RC filter

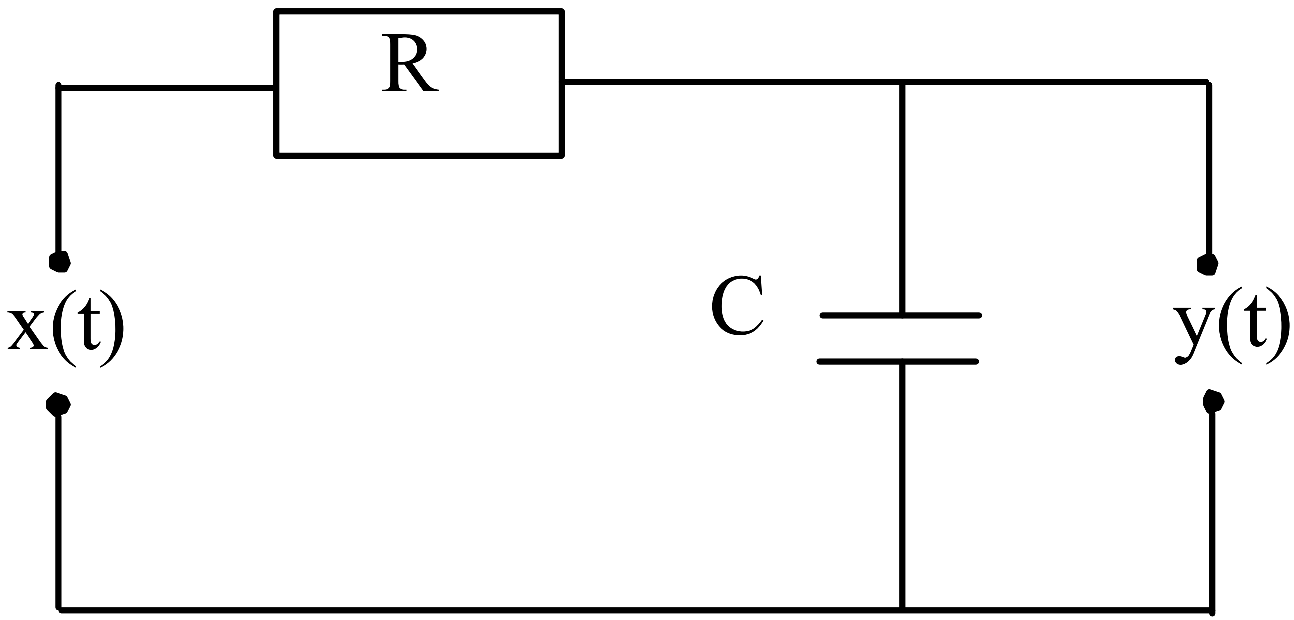
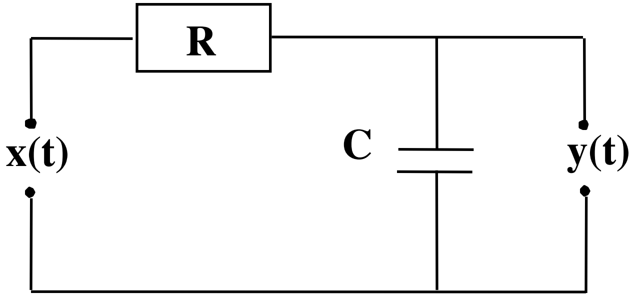


Fig. 2.1 RC filter.

Differential equation



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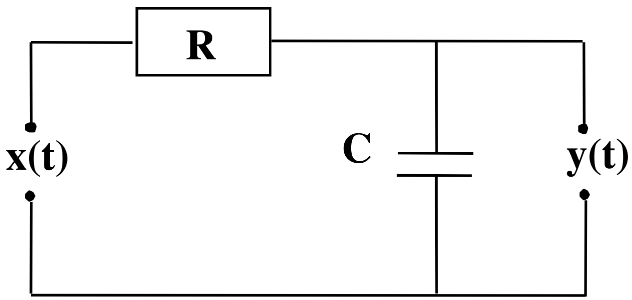
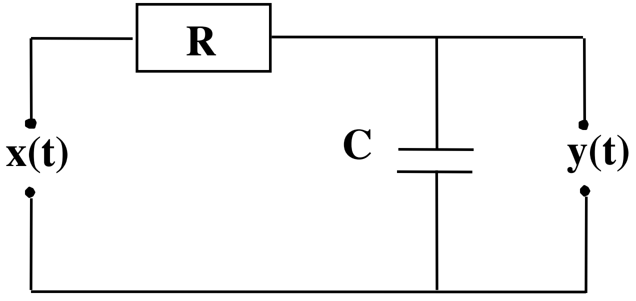


Voltage balance?

Differential equation



Voltage balance?

$$R I(t) + y(t) = x(t)$$

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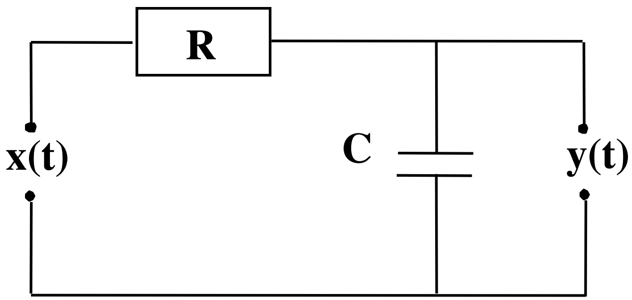

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Current through resistor R and capacitance C:

$$I(t) = C\dot{y}(t)$$

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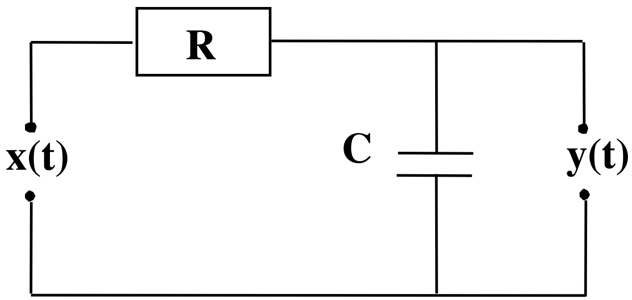

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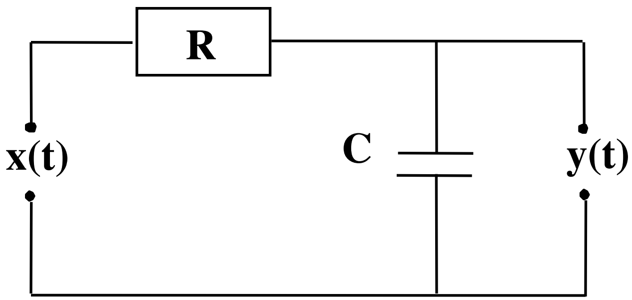

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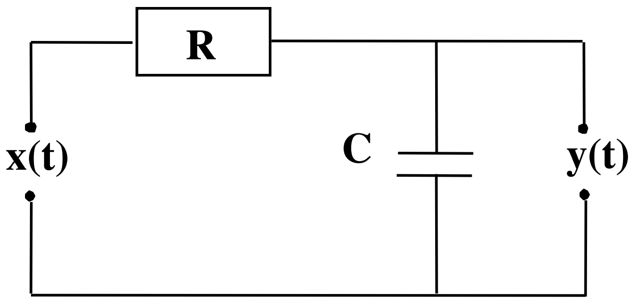

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First order linear differential equation

Linearity

Input **Output**

Linearity

Input **Output**

$x_1(t) \longrightarrow y_1(t)$

Linearity

Input		**Output**
$x_1(t)$	$\longrightarrow$	$y_1(t)$
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RC filter = linear (time invariant) system = LTI system

Frequency response function and Fourier transform

Frequency response function and Fourier transform

$$RC\dot{y}(t) + y(t) - x(t) \;=\; 0 \qquad \text{for zero input signal x(t)}$$

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Approach: Solve for harmonic signals,
then construct arbitrary signals from harmonic signals.

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Back transformation: $$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$

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($\Leftrightarrow$ indicates a transform pair, $x(t) \Leftrightarrow X(j\omega)$):

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- *Derivative —* $\dfrac{d}{dt}x(t) \Leftrightarrow j\omega \cdot X(j\omega)$
- *Integration —* $\displaystyle\int_{-\infty}^{\infty} x(t)dt \Leftrightarrow \frac{1}{j\omega} \cdot X(j\omega)$

Convolution

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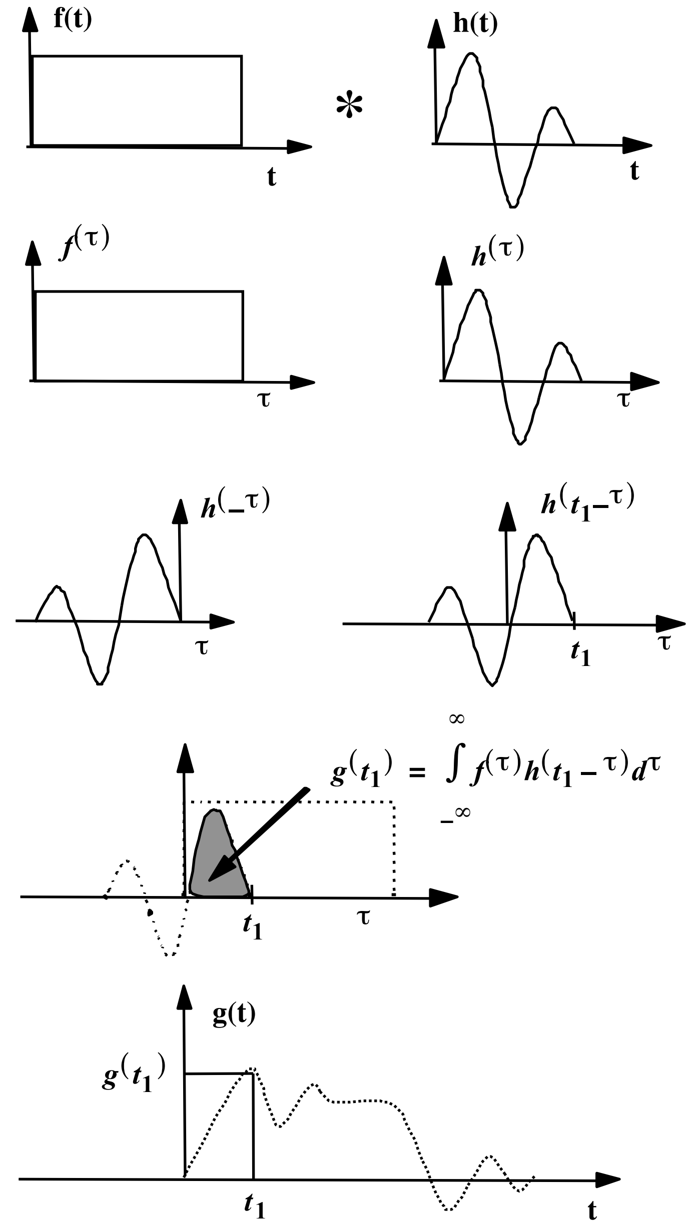


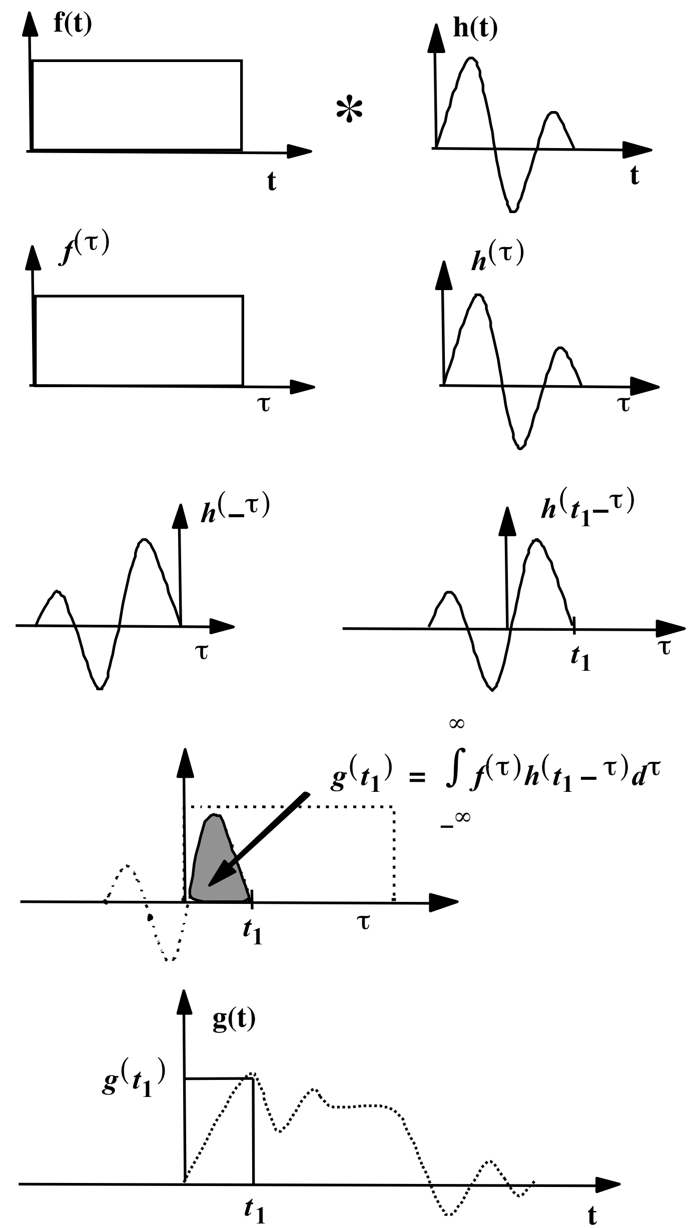
Fig. 2.5 Graphical interpretation of the convolution operation.

Convolution

$$g(t) = f(t) * h(t) = \int_{-\infty}^{\infty} f(\tau) \cdot h(t - \tau) d\tau$$

- *Convolution Theorem —*

$$\mathrm{f(t)} * \mathrm{h(t)} \Leftrightarrow \mathrm{F(j\omega)} \cdot \mathrm{H(j\omega)}$$



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Harmonic input signals

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Frequency response function

Input/output relation

$$\boxed{A_o = T(j\omega) \cdot A_i}$$

Polar form

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$$T(j\omega) = \frac{1}{RCj\omega + 1} \longrightarrow \text{polar form}$$

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Identities: $$\frac{1}{\alpha + j\beta} = \frac{\alpha}{\alpha^2 + \beta^2} - j\frac{\beta}{\alpha^2 + \beta^2}$$ and $$\Phi = \arctan\left(\frac{\mathrm{Im}}{\mathrm{Re}}\right)$$

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and $T(j\omega) = \frac{1}{\sqrt{1 + (RC\omega)^2}} \cdot e^{j\Phi(\omega)}$

$$\Phi(\omega) = \arctan(-RC\omega) = -\arctan(RC\omega)$$

The frequency response and arbitrary input signals

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$A_i(j\omega) \;\rightarrow\;$ harmonic component of the Fourier spectrum $X(j\omega)$ (input)

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• ***Definition*** **—** The frequency response function $T(j\omega)$ is defined as the Fourier transform of the output signal divided by the Fourier transform of the input signal.

Input/output relation

Fourier spectrum of the filter output:

$$Y(j\omega) = T(j\omega) \cdot X(j\omega)$$

The frequency response function can be measured by comparing output and input signals to the system **without further knowledge of the physics going on inside the filter**!

Transfer function and Laplace transform

Bilateral Laplace transform of *f(t)*: $\mathbf{L}\left[f(t)\right] = \int_{-\infty}^{\infty} f(t)e^{-st}dt$

with the complex variable $s = \sigma + j\omega$

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Property: $\mathcal{L}\left[\dot{f}(t)\right] = s \cdot F(s)$

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Transforming equation $RC\dot{y}(t) + y(t) - x(t) = 0$

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Transfer function

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a) $s \rightarrow j\omega \;\Rightarrow\; T(s) \rightarrow T(j\omega)$

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RC Filter:

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Special cases:

a) $s \to j\omega \;\Rightarrow\; T(s) \to T(j\omega)$

b) $s \to -\frac{1}{\tau} \;\Rightarrow\; T(s) \to \infty$

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RC Filter:

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Special cases:

a) $s \rightarrow j\omega \;\Rightarrow\; T(s) \rightarrow T(j\omega)$

b) $s \rightarrow -\frac{1}{\tau} \;\Rightarrow\; T(s) \rightarrow \infty$ **"pole"**

The impulse response function

Dirac delta 'function' $\delta(t)$ $$\int_{-\infty}^{\infty} f(t)\delta(t-\tau)dt = f(\tau)$$

The impulse response function

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Generation:

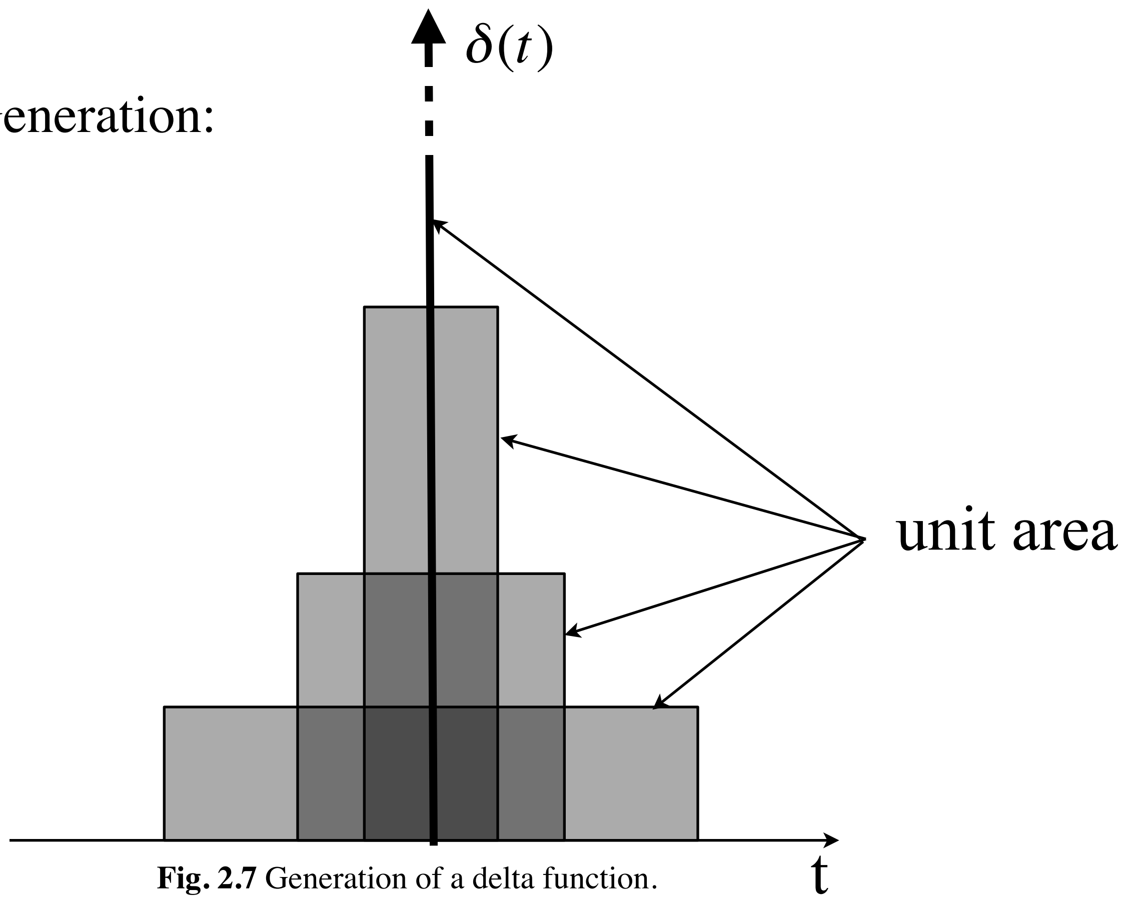

Fig. 2.7 Generation of a delta function.

From the conditions of unit area and infinitesimal duration we obtain

$$\int_{-\infty}^{\infty} \delta(t)dt = 1$$

and $\quad \delta(t) = 0 \quad$ for $\quad t \neq 0$

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Properties $\quad \mathbf{F}\left\{\delta(t)\right\} = \int_{-\infty}^{\infty} \delta(t) \cdot e^{-j2\pi ft} dt = 1$

$$\mathbf{L}\left\{\delta(t)\right\} = \int_{-\infty}^{\infty} \delta(t) \cdot e^{-st} dt = 1$$

The impulse response function

Response of a filter to an impulsive (delta function) input signal.

Properties:

- The frequency response function $T(j\omega)$ is the Fourier transform of the impulse response function.

- The transfer function $T(s)$ is the Laplace transform of the impulse response function.

Fourier spectrum of a filter output signal:

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Convolution theorem:

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Convolution theorem:

$$T(j\omega) \cdot X(j\omega) = H(j\omega) \cdot X(j\omega) \quad \Leftrightarrow \quad h(t) * x(t)$$

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Consequences:

Filtering

$$\boxed{\begin{aligned} y(t) &= h(t) * x(t) \\ Y(j\omega) &= T(j\omega) \cdot X(j\omega) \\ Y(s) &= T(s) \cdot X(s) \end{aligned}}$$

The frequency response function and the pole position

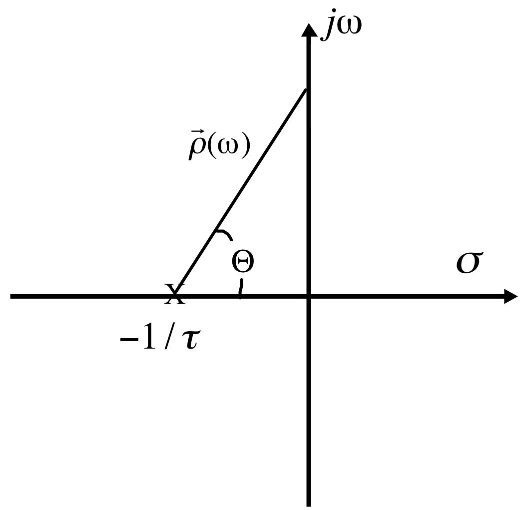


Fig. 2.10 Representation of the RC filter in the s plane. The pole location at -1/τ is marked by an X.

Transfer function RC filter: $$T(s) = \frac{1}{1+s\tau} = \frac{1}{\tau}\left[\frac{1}{(1/\tau)+s}\right]$$

For $s = j\omega$, ω moves along the imaginary axis

$1/\tau + j\omega$ represents the vector $\vec{\rho}(\omega)$

$$\Longrightarrow \quad T(j\omega) = \frac{1}{\tau}\left[\frac{1}{(1/\tau)+j\omega}\right]$$

which is pointing from the pole position towards the actual frequency on the imaginary axis.

in polar coordinates

$$T(j\omega) = \frac{1}{\tau}\left[\frac{1}{\left|\vec{\rho}(\omega)\right| e^{j\Theta}}\right] = \frac{1}{\tau}\left[\frac{1}{\left|\vec{\rho}(\omega)\right|} e^{-j\Theta(j\omega)}\right] = \left|T(j\omega)\right| e^{j\Phi(j\omega)}$$

For the given example, the amplitude value of the frequency response function for frequency ω is proportional to the reciprocal of the length of the vector $\vec{r}(\omega)$ from the pole location to the point $j\omega$ on the imaginary axis. The phase angle equals the negative angle between $\vec{r}(\omega)$ and the real axis.

Problem 2.2

Determine graphically the amplitude characteristics of the frequency response for a RC filter with $R = 4.0$ Ohm and $C = (1.25/2\pi)\ F = 0.1989495\ F$ (1Ohm = 1(V/A), 1F = 1As/V). Where is the pole position in the S plane? For the plot use frequencies between 0 and 5 Hz.

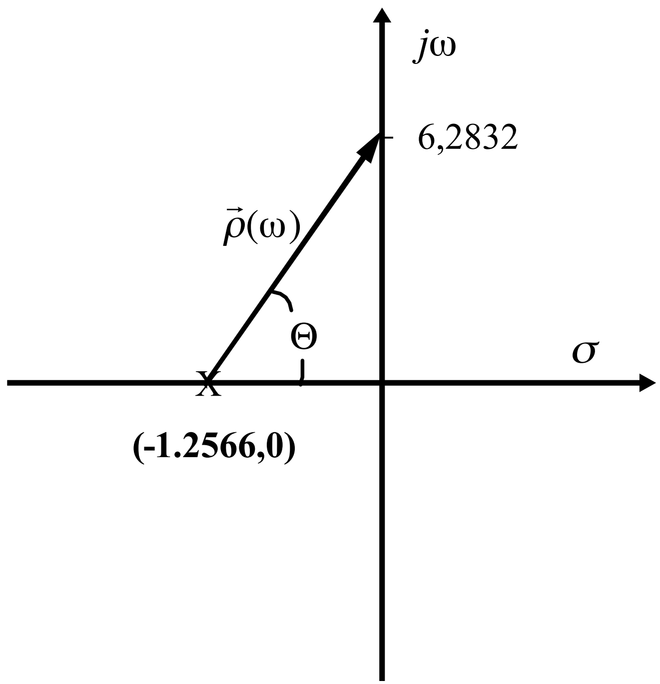
Problem 2.3

Calculate the frequency response for the RC filter from Problem 2.2 using the Digital Seismology Tutor.

Start up: Digital Seismology Tutor

Solution 2.2

Solution 2.2 The pole position is at $-1/\tau$. $\tau = R\ C$ = 4.0 Ohm 0.1989495F which is 4.0 V/A 0.1989495$A sec/V$ = 0.795798sec. Hence, the pole is at –1.2566 (rad/s). For each point on the imaginary axis (angular frequency axis), determine the reciprocal of the length of the vector from the pole to that point. You can do this either by using a ruler and graph paper or simply by exploiting analytical geometry. Plot this value as a function of angular frequency or frequency, respectively. Below, the procedure is demonstrated schematically for a frequency of 1Hz (Fig. A 2.1; note, that Fig. A 2.1 is not on 1:1 scale).



Example: f=1 Hz

$$\omega = 2\pi \cdot 1 = 6{,}2832$$

$$\left|\vec{\rho}(2\pi)\right| = 6{,}4076$$

$$\left|T(j2\pi)\right| = \frac{1}{\tau}\left[\frac{1}{\left|\vec{\rho}(\omega)\right|}\right]$$

$$\left|T(j2\pi)\right| = \frac{1{,}2566}{6{,}4076} = 0{,}1961$$

Fig. A 2.1 Graphical determination of the modulus of the frequency response function for the RC filter of Problem 2.2. The plot demonstrates the evaluation for a frequency of 1 Hz.

Solution 2.3

Solution 2.3 The Digital Seismology Tutor (DST) simulates the action of systems defined by their transfer function in the complex s-plane. As we will see later in more detail, a transfer function of a more general system can have more than one pole as well as a number of zeros (at which the transfer function becomes zero). The positions of poles and zeros define the transfer function completely. In order to do the filtering, DST needs to know the position(s) of the pole(s) and zero(s) (which will be introduced in later chapters) in the complex s-plane.

After starting up the DST, select the *Modify and Enter* option from the *Poles/Zeros* menu. In order to enter the pole position for the RC-filter at (-1.2566, 0), enter the real part (-1.2566) into the uppermost left box and the imaginary part (0.0) into the uppermost right box as shown in Fig. A 2.2. Enter a 1.0 for the scale factor in the bottom box. Finally accept the input either by using the *OK* or the *Apply* button. The difference between these two is merely that the *OK* button closes the window after accepting the input while the *Apply* button leaves is open for further input.

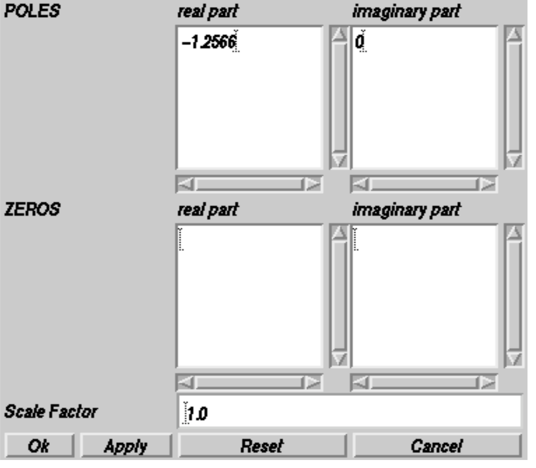


Fig. A 2.2 How to enter the pole position for Problem 2.3 into the *Modify and Enter Poles/ Zero* panel of the DST.

Solution 2.3, cont. 1

After you entered the pole into the DST, you can visualize its position within the complex s-plane by using the *Map* option from the *Poles*/*Zeros* menu (Fig. A 2.3).

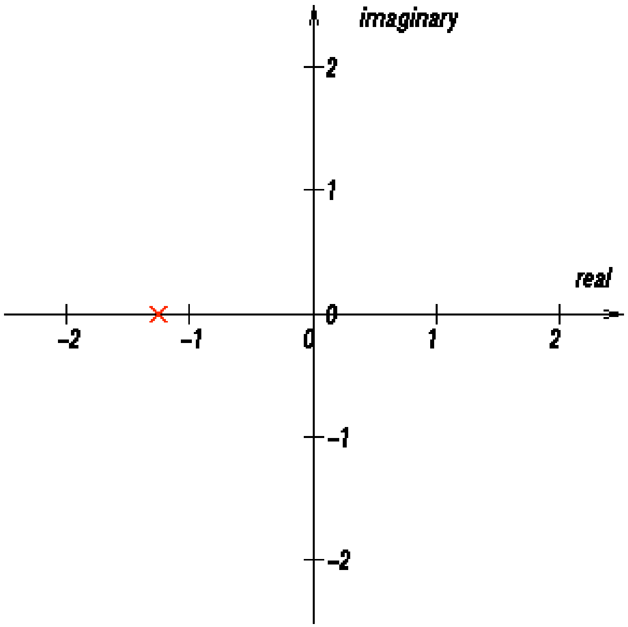


Fig. A 2.3 Pole position of RC-filter from Problem 2.3 within the complex s-plane as calculated with DST.

Solution 2.3, cont. 2

The corresponding frequency response can be viewed using the *Show Response* menu from the DST main window using the option *Frequency Response -> Amplitude-> lin-lin* (Fig. A 2.4).

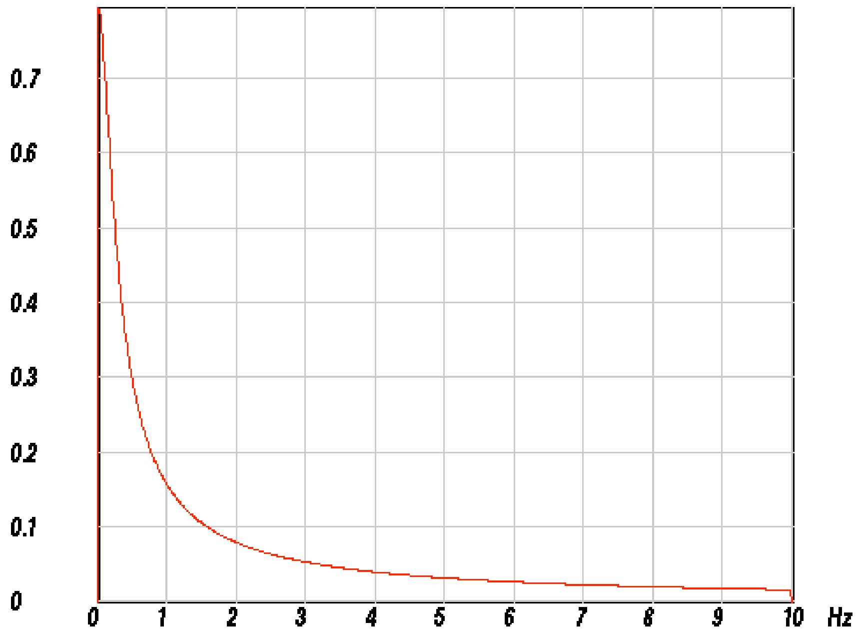


Fig. A 2.4 Frequency response function (amplitude only) for the pole position shown in Fig. A 2.3.

Note: Although the plot in Fig. A 2.4 shows a continuous curve, it is actually a discrete approximation of the continuous frequency response function. The details of the underlying relationship will be explained in detail in chapter 7 “From infinitely continuous to finite discrete”. At this point it is sufficient to know that the internal sampling frequency for the calculation of the frequency response function can be modified in the *Setup* menu. For reasons which are explained in chapter 7, the frequency band which is plotted ranges from 0 to 1/2 of the internal sampling frequency.

Lin-Lin scale

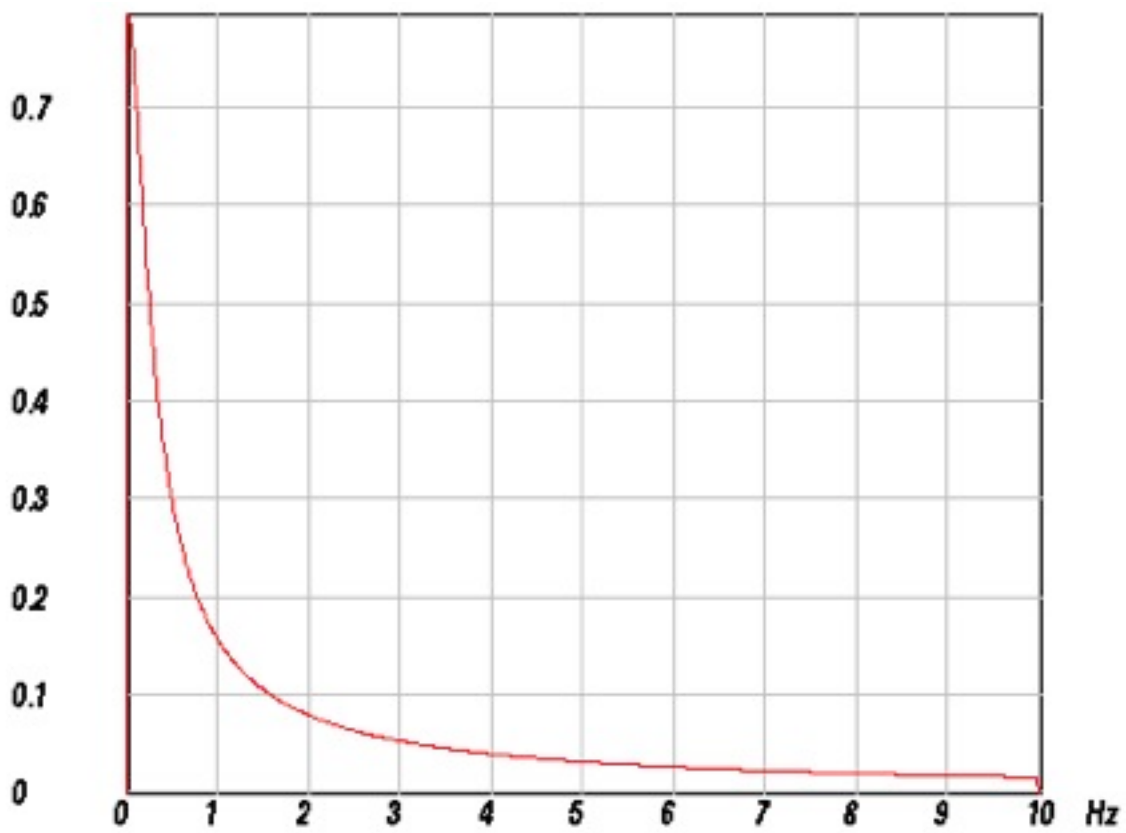


Fig. 2.11 Frequency response function (amplitude only) of the RC filter of Problem 2.3.

Log-Log scale

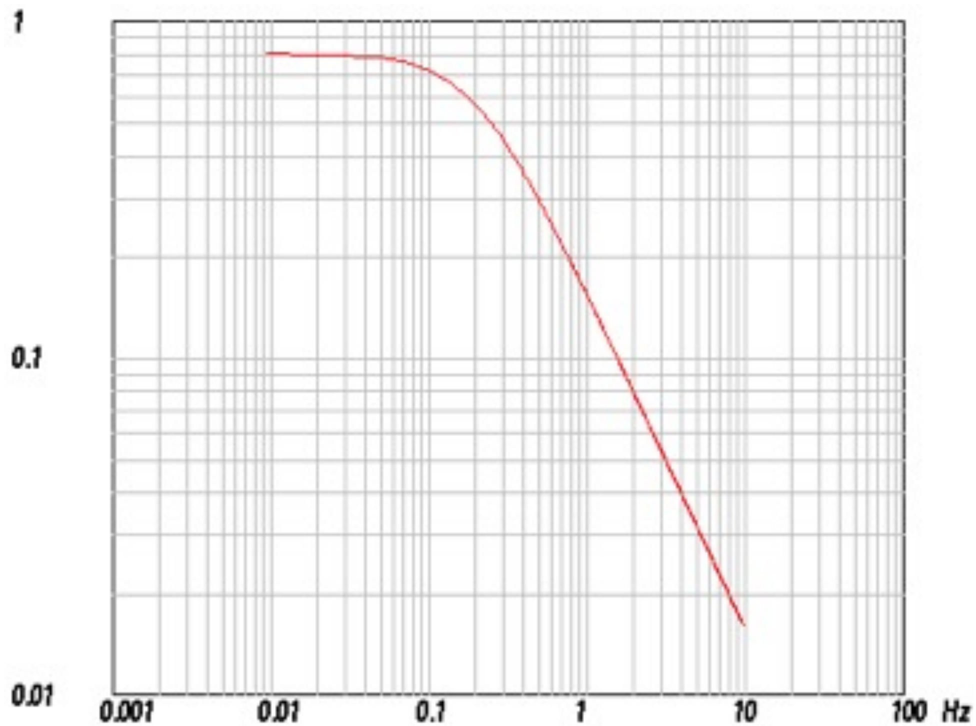


Fig. 2.12 Same plot as Fig. 2.11 only on a log-log scale.

Log-Log scale

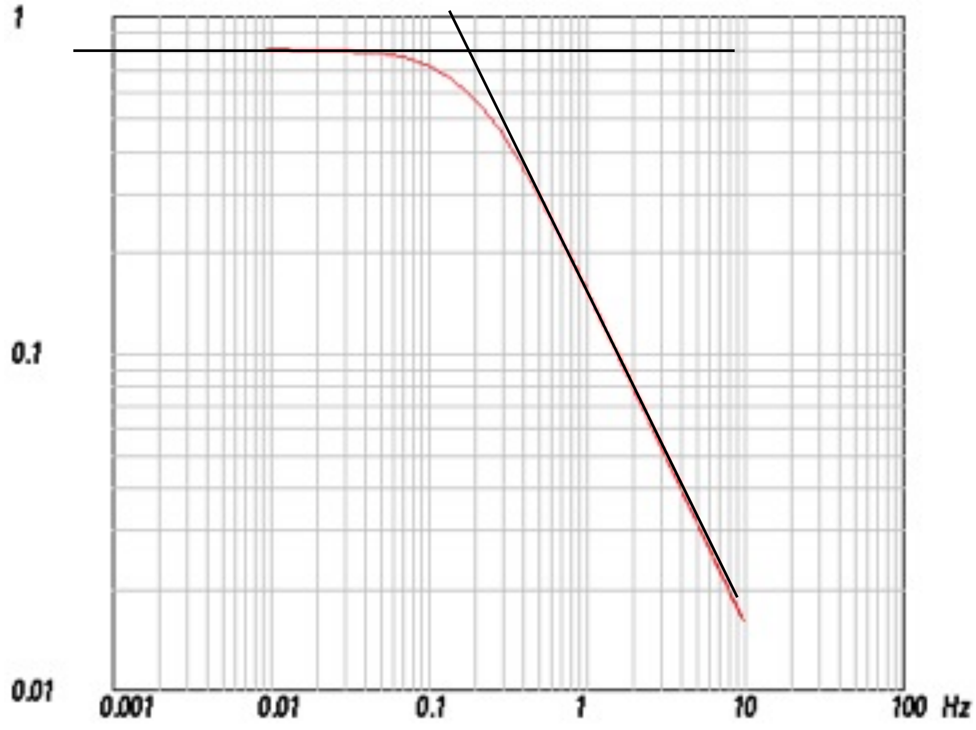


Fig. 2.12 Same plot as Fig. 2.11 only on a log-log scale.

Log-Log scale

Corner frequency:
$0{,}2Hz = 1/5 \cdot \sec^{-1} = 1/(RC \cdot 2\pi)$

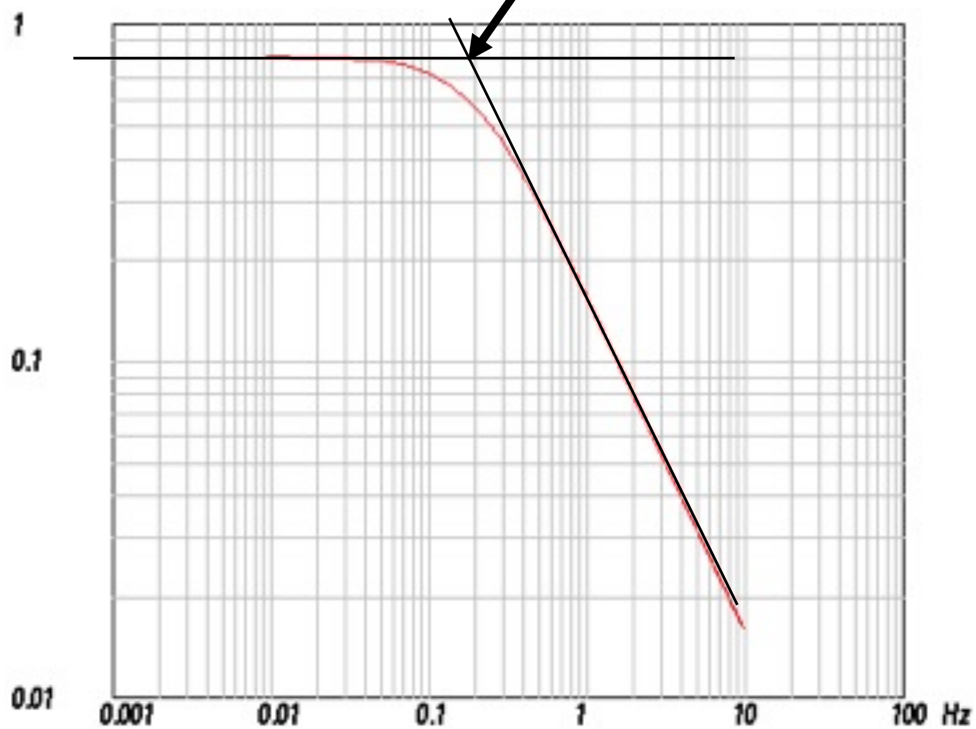


Fig. 2.12 Same plot as Fig. 2.11 only on a log-log scale.

Shape of frequency response function

$$\left[\text{Corner frequency: } 0{,}2Hz = 1/5 \cdot \sec^{-1} = 1/(RC \cdot 2\pi)\right]$$

$$\left|T(j\omega)\right| = \frac{1}{\tau}\left[\frac{1}{\left|(1/\tau) + j\omega\right|}\right] = \frac{1}{\sqrt{1+\omega^2\tau^2}}$$

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Definition: $\omega_c := \left|-\frac{1}{\tau}\right| \quad \Longrightarrow \quad |T(j\omega)| = \frac{1}{\sqrt{1+\frac{\omega^2}{\omega_c^2}}}$

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$$\omega \to 0: \quad |T(j\omega)| \to 1 = const$$

$$\omega \gg \omega_c: \quad |T(j\omega)| \to \omega^{-1}$$

$$Slope_{\log-\log} = \frac{\log_{10} A(\omega_2) - \log_{10} A(\omega_1)}{\log_{10}(\omega_2) - \log_{10}(\omega_1)} = \frac{\log_{10}\left(\frac{A(\omega_2)}{A(\omega_1)}\right)}{\log_{10}\left(\frac{\omega_2}{\omega_1}\right)}$$

Amplitude ratio in *dB* (20 $\log_{10}$(amplitude ratio)):

$$Slope_{dB/D\omega} = 20 \cdot \frac{\log_{10}\left(\dfrac{A(\omega_2)}{A(\omega_1)}\right)}{\log_{10}\left(\dfrac{\omega_2}{\omega_1}\right)}$$

for $|T(j\omega)| >> \omega^{-1}$

amplitude decreases by a factor of 10 over a full decade

Therefore $Slope_{dB/dec} = 20 \cdot \dfrac{\log_{10}(0{,}1)}{\log_{10}(10)} = -20dB \quad [dB\,/\,decade]$

or following the same argument -6 *dB/octave*

General rule:

> Rule: A single pole in the transfer function causes the slope of the amplitude portion of the frequency response function in a log-log plot to decrease by 20 dB/decade or 6 dB/octave, respectively.

The RC filter and the role of the pole

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Transfer function: $T(s) = \dfrac{1}{1 + s\tau}$ pole: $s_p = -\dfrac{1}{\tau}$

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The RC filter and the role of the pole

Transfer function: $T(s) = \dfrac{1}{1 + s\tau}$ pole: $s_p = -\dfrac{1}{\tau}$

- **determines boundary of ROC**
- **position determines stability**
- **length of pole vector determines magnification**

$$|T(j\omega)| \sim \left|\frac{1}{\rho(\omega)}\right|$$

The RC filter and the role of the pole

Transfer function: $T(s) = \dfrac{1}{1 + s\tau}$ pole: $s_p = -\dfrac{1}{\tau}$

- determines boundary of ROC
- position determines stability
- length of pole vector determines magnification

$$|T(j\omega)| \sim \left|\frac{1}{\rho(\omega)}\right|$$

- a pole in the transfer function changes the slope of the modulus of the frequency response function by ω^{-1} (20 dB/dec, 6dB/oct) at a corner frequency $\omega_c = |s_p|$