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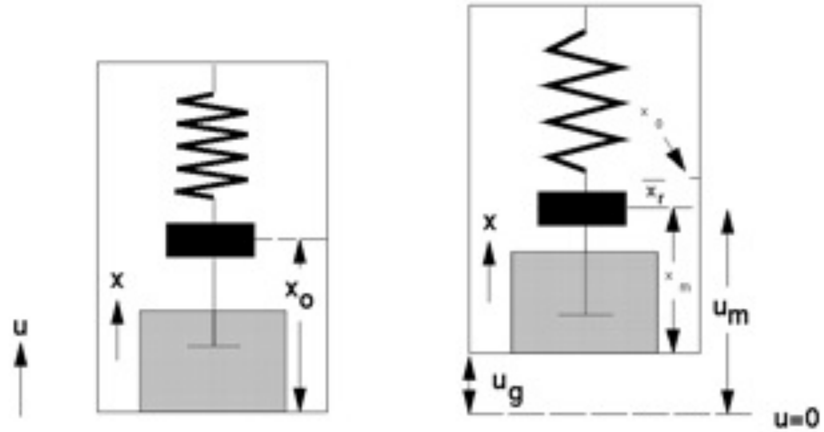
WHAT IS IT GOOD FOR????

Seismometer



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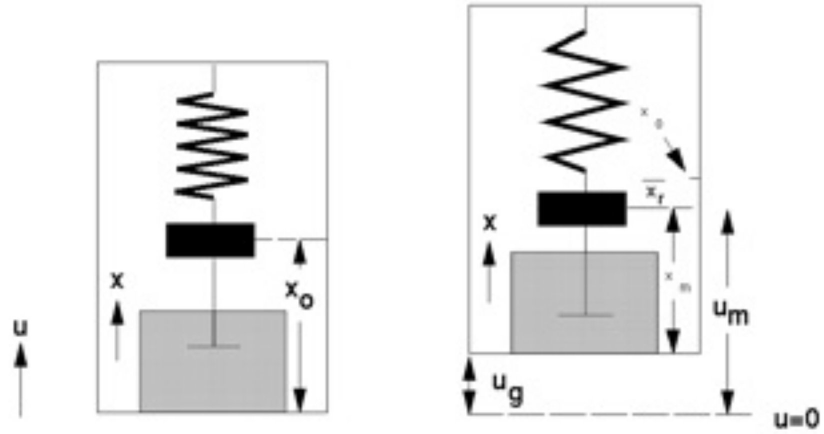

• *The inertia of the mass –* $f_i = -m\ddot{u}_m(t)$

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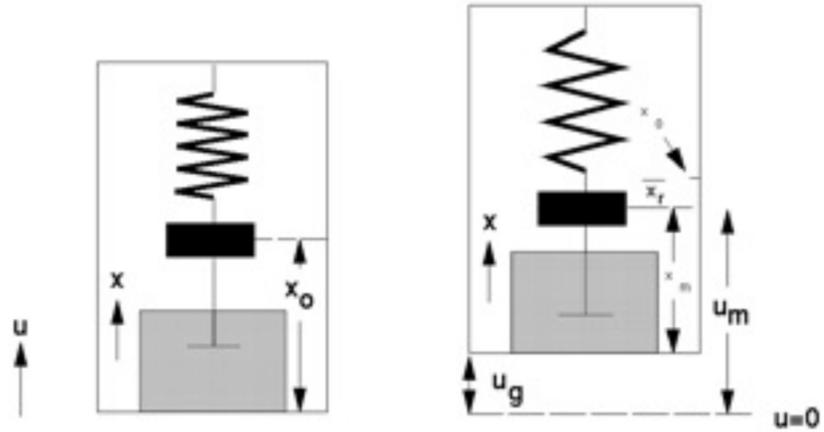

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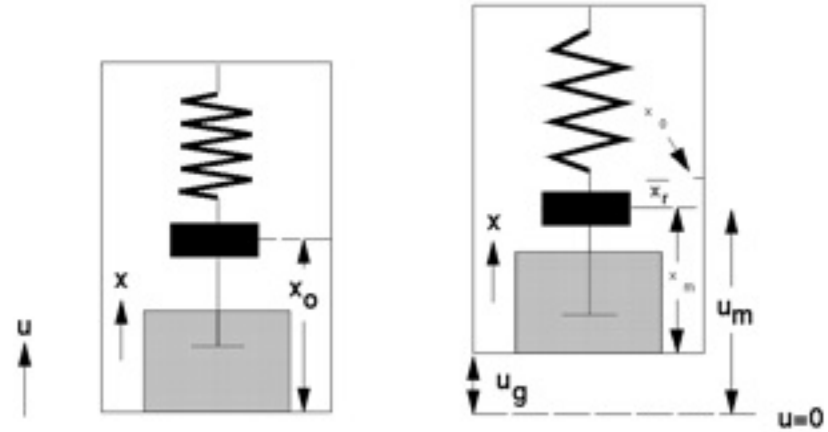

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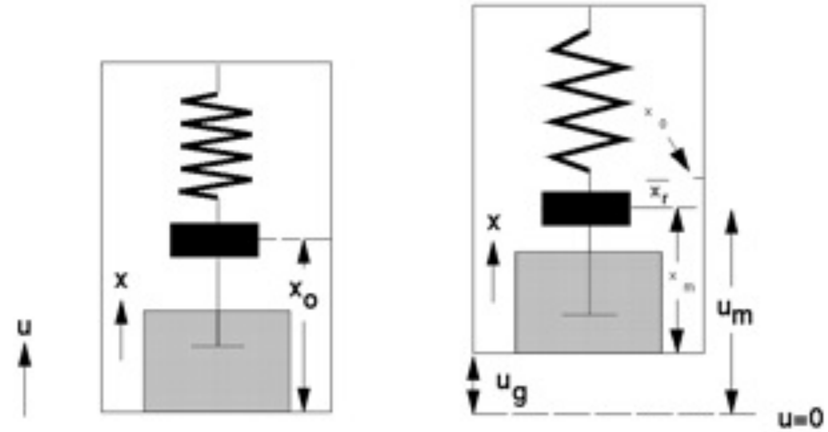

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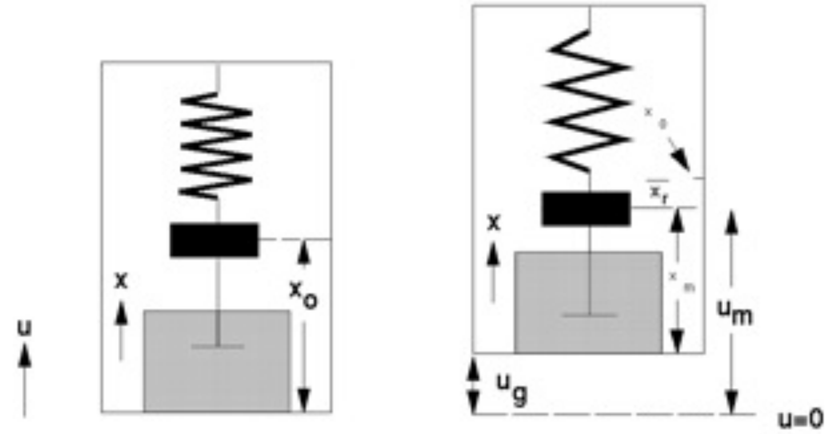

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2nd order LTI !!

Problem 3.1

System with two poles. Consider three different cases:

a) Put both poles at -1.2566, 0.
b) Put one pole at location -1.2566, 0 and the other one at 1.2566, 0.
c) Put both poles at 1.2566, 0.

For the input signal, use a spike at the center position of the window (for DST an internal sampling frequency of 100Hz and a window length of 2048 points works well). What types of impulse response functions do you expect in each case? Will the frequency response functions be different? What changes do you expect for the frequency response functions with respect to Problem 2.3 (single pole at –1.2566,0)?

Consequences of transition from single pole to general N-th order system

- No major change in concept
- Zeros in addition to poles
- Can be treated in very similar way

Graphical estimation of the frequency response function

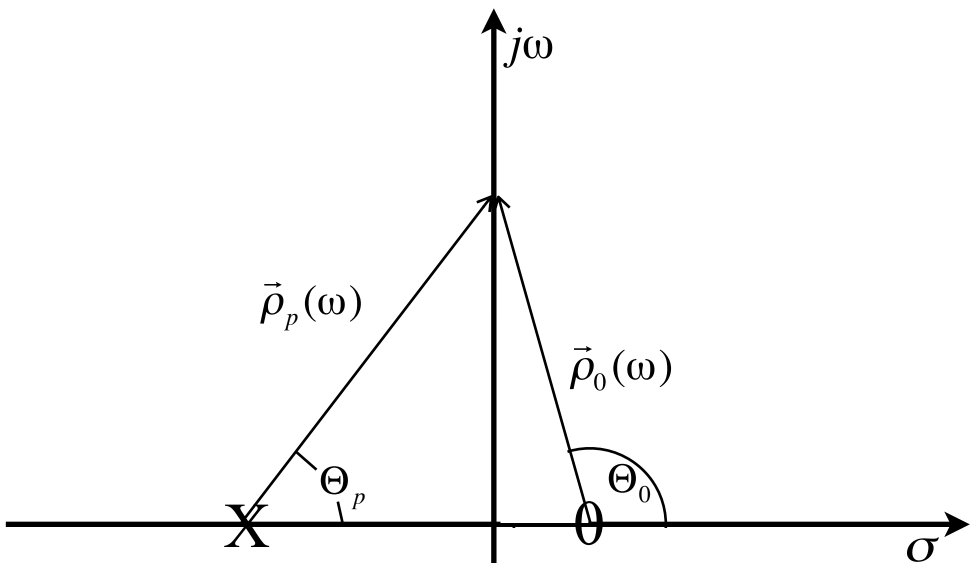

Fig. 3.2 Complex s plane representation of a system with a single pole and zero. The pole and zero locations are marked by an X, and a 0, respectively.

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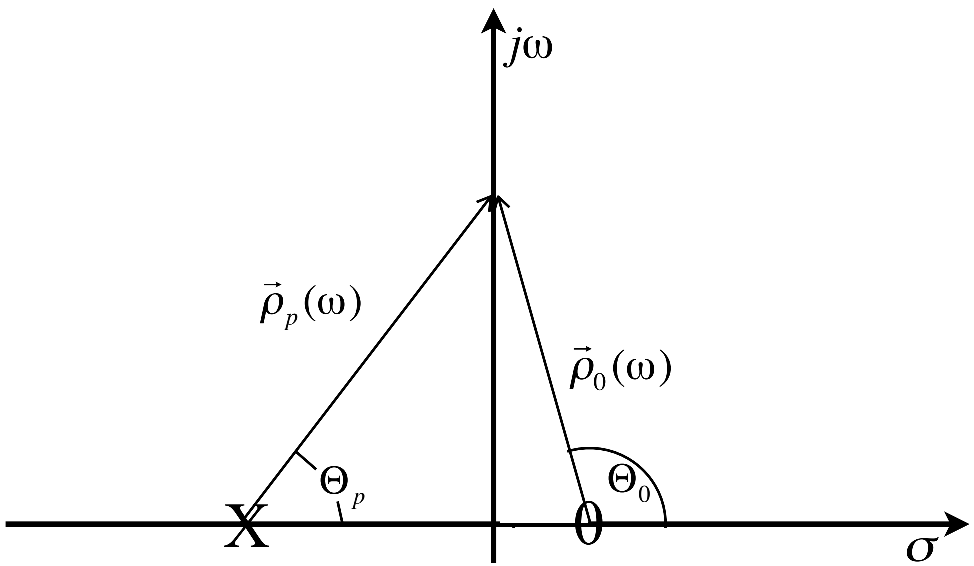

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Transfer function: $$T(s) = \frac{s - s_0}{s - s_p}$$

s_0 and s_p: position of the zero and the pole, respectively.

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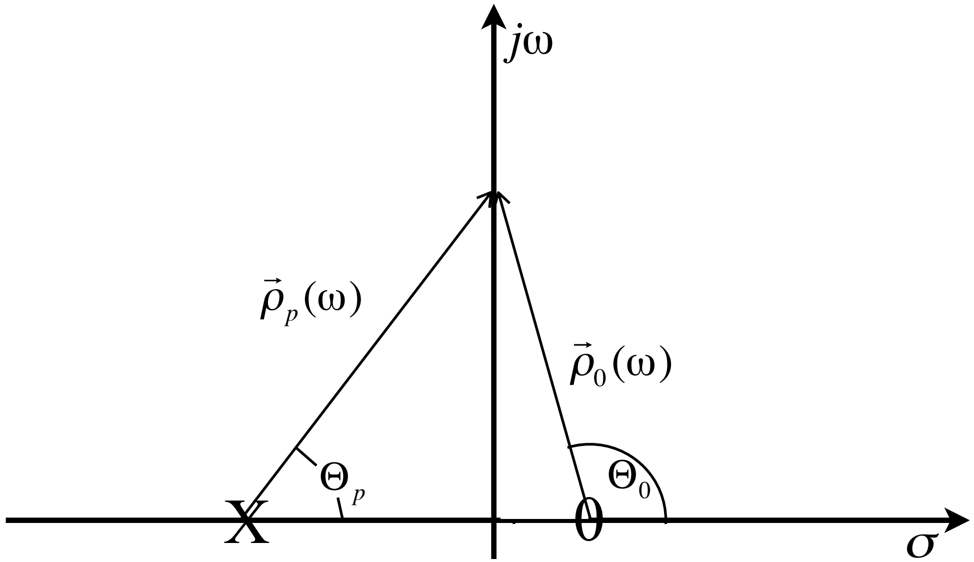

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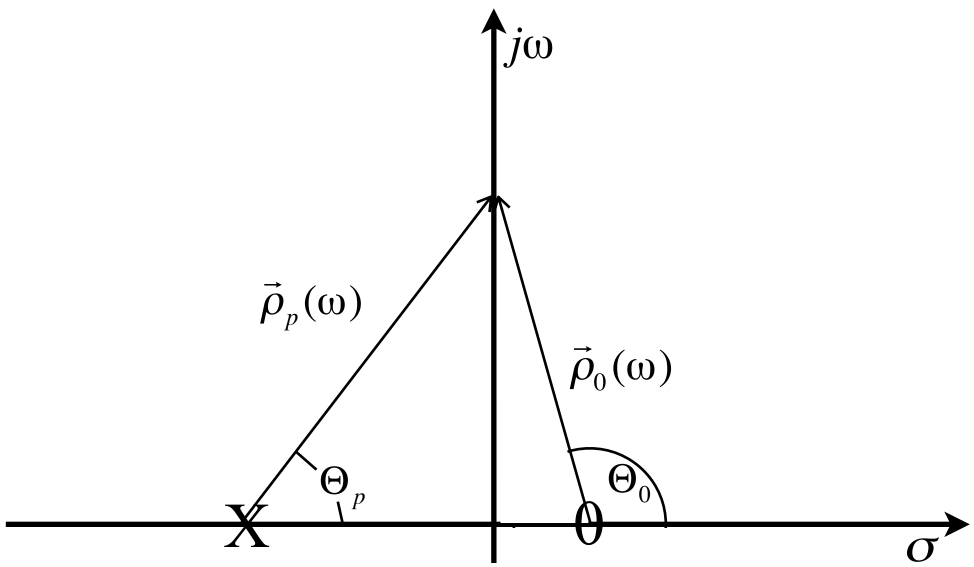

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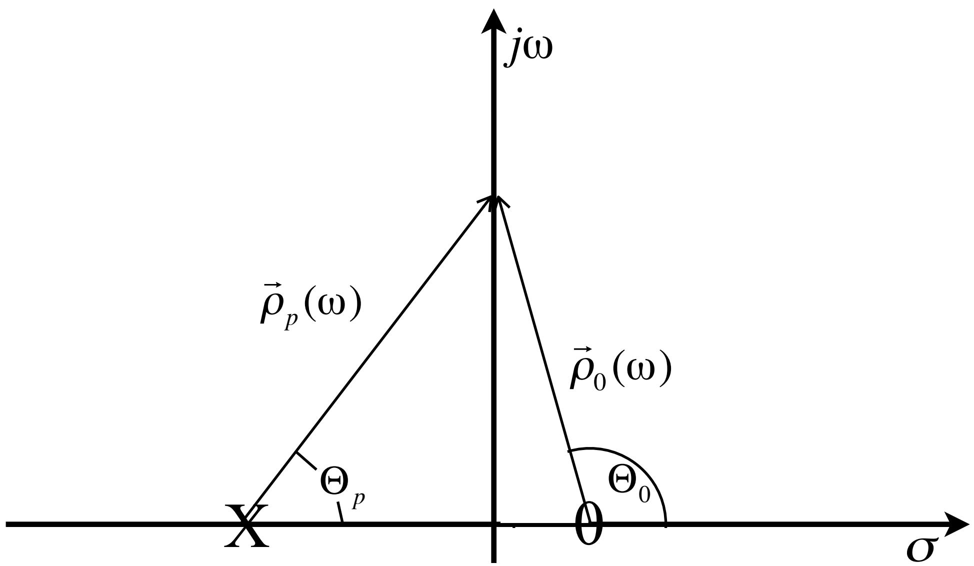

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$$T(j\omega) = \left|\rho_0(\omega)\right| e^{j\Theta_0} \frac{1}{\left|\rho_p(\omega)\right|} e^{-j\Theta_p}$$

Arbitrary LTI system

The amplitude part of the frequency response function of an arbitrary LTI system can be determined graphically by multiplying the lengths of the vectors from the zero locations in the S plane to the point $j\omega$ on the imaginary axis divided by the product of all lengths of vectors from pole locations to the point $j\omega$ on the imaginary axis. Likewise, to determine the phase part, the phase angles for the vectors from the zero locations in the S plane to the point $j\omega$ on the imaginary axis have to be added together. Then, the phase angles of all the vectors from pole locations to the point $j\omega$ on the imaginary axis have to be subtracted.

Problem 3.2

Use the argument given above to determine the frequency response for a system with a single pole at –1.2566,0 if you add a zero at position 1.2566, 0?

The phase properties of general LTI system

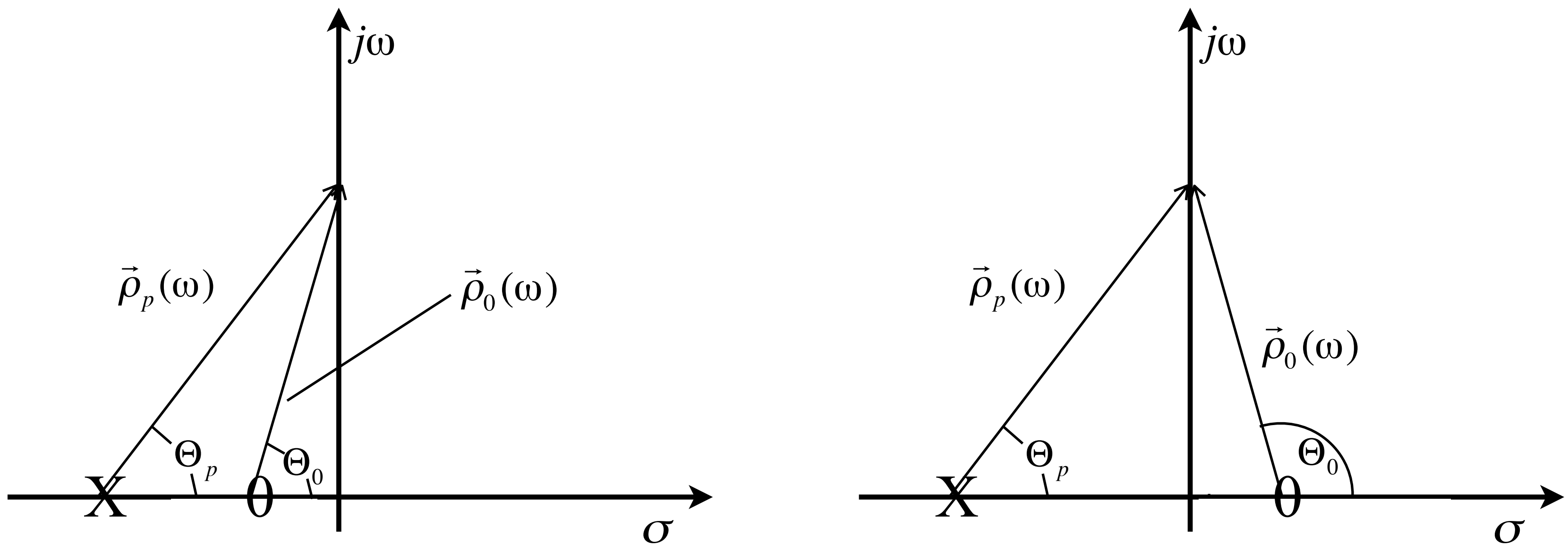

Fig. 3.3 Complex s plane representation of two systems with a single pole and zero. In b) the zero is at the same distance from the origin as in a) The pole and zero locations are marked by an X, and a 0, respectively.

How does the amplitude response differ?

How do the phase properties differ?

Minimum/maximum phase

A **causal stable** system (no poles in the right half plane) is minimum phase provided it has no zeroes in the right hand plane.

It is maximum phase if it has all its zeroes in the right hand plane.

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minimum phase systems: nice properties !

Problem 3.3

How can the following two statements be proven for a general LTI system? a) If a system is minimum phase it will always have a stable and causal inverse filter. b) Any mixed phase system can be seen as a convolution of a minimum phase system and an allpass filter, which only changes the phase response but leaves the amplitude response as is.

Problem 3.4

How can we change the two-sided impulse response from Problem 3.1b (one pole at –1.2566,0 and another one at 1.2566,0) into a right-sided one without changing the amplitude response? Keyword: allpass filter.

The interpretation of the frequency response function

A single pole in the transfer function causes the slope of the amplitude frequency response function to decrease by 20 dB/decade (6 dB/octave).

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The transition in either case occurs at corner frequencies which are equal to the modulus of the pole/zero position.

General rule

Each pole in the transfer function causes the slope of the amplitude frequency response function to decrease by 20 dB/decade (6 dB/octave).

Each zero causes an increase of the slope by the same amount .

The transition in either case occurs at corner frequencies which are equal to the moduli of the pole/zero positions.

Problem 3.5

Consider a system with a pole and a zero on the real axis of the s plane. Let the pole position be (-6.28318, 0), and the zero position (.628318,0). What is the contribution of the zero to the frequency response function?

An internal sampling frequency of 20 Hz is recommended in DST

One more point...

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Are we allowed to do this?

Yes, but: Real systems require conjugate complex singularities.

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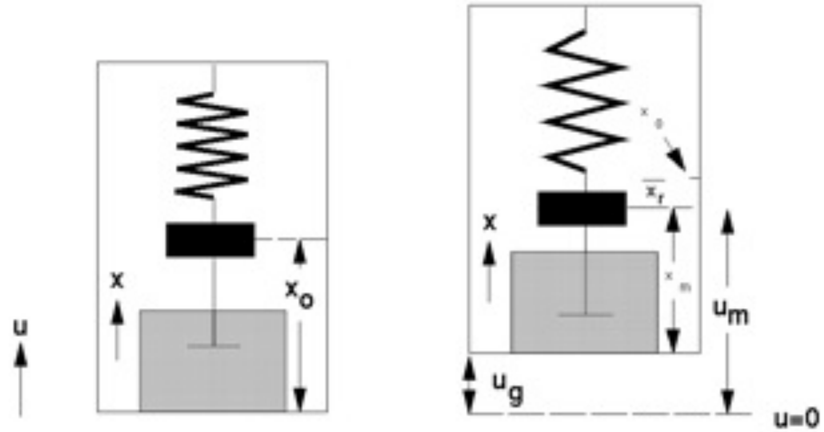

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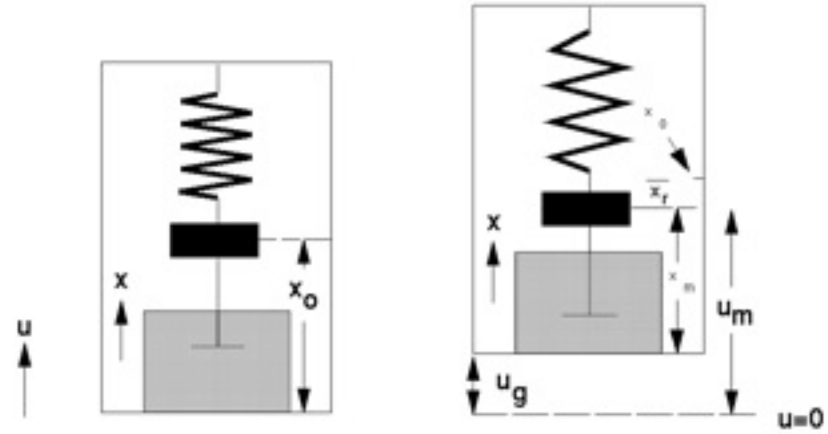

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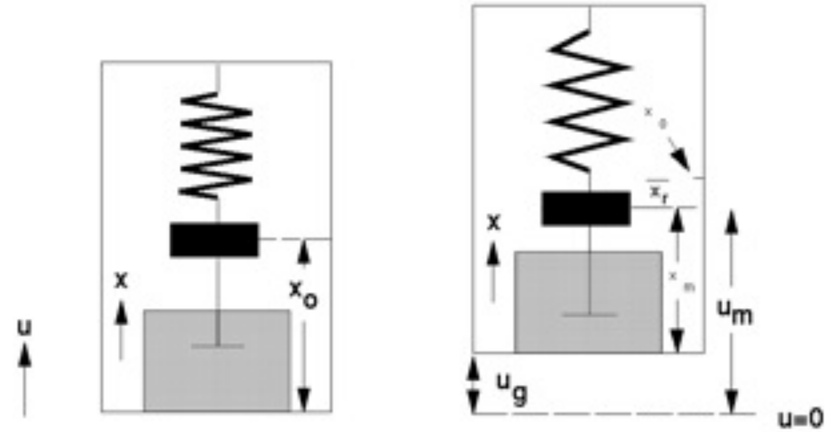

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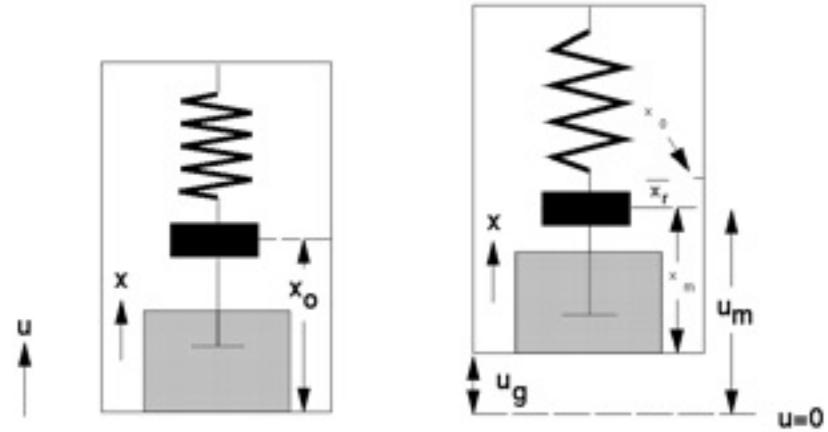

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Since a quadratic equation $x^2 + bx + c = 0$ has the roots

$$x_{1,2} = -\frac{b}{2} \pm \sqrt{\frac{b^2}{4} - c}$$

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pole positions $p_{1,2}$:

$$
\begin{aligned}
p_{1,2} &= -\varepsilon \pm \sqrt{\varepsilon^2 - \omega_0^2} \\
&= -h\omega_0 \pm \omega_0 \sqrt{h^2 - 1} \\
&= -\left(h \pm \sqrt{h^2 - 1}\right)\omega_0
\end{aligned}
$$

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Therefore, the poles of an underdamped seismometer are located in the left half of the s plane in a distance of $|\omega_0|$ from the the origin. The quantity $h\,|\omega_0|$ gives the distance from the imaginary axis.